JOURNAL OF APPROXIMATION THEORY 1, 374–380 (1968)

Nonnegative Interpolation Formulas for Uniformly Elliptic Equations[1]

P. J. DAVIS

Brown University, Providence, Rhode Island 02912

AND

M. W. WILSON

IBM Watson Research Center, Yorktown Heights, New York 10598

In this paper, we show the existence of nonnegative interpolation formulas for functions which are solutions of second-order uniformly elliptic equations over bounded domains.

1. INTRODUCTION

In this paper we are interested in the existence of nonnegative interpolation formulas for functions which are solutions of second-order uniformly elliptic differential equations over bounded domains in E^r. The existence theory is based on properties of nonnegative linear functionals. Some of this theory is outlined in Section 2, and extended in a direction useful for interpolation. The theory is applied to solutions of elliptic equations in Section 3.

2. MOMENT CONE STRUCTURE

Let T be a compact set in E^r, and let $C_n(T)$ be the $(n+1)$-dimensional linear vector space spanned by the linearly independent continuous functions $\varphi_0(t)$, $\varphi_1(t)$, ..., $\varphi_n(t)$ defined on T. We assume the *Krein Condition* (see Rogosinski [8]) is satisfied, namely, $\exists\, p(t) \in C_n(T)$ such that $p(t) \geqslant \alpha > 0$ on T.

Relative to the basis φ_0, φ_1, ..., φ_n of $C_n(T)$, we can represent any linear functional L in the dual space $C_n(T)^*$ by its *moment vector* $(L\varphi_0, L\varphi_1, \ldots, L\varphi_n)^t$, a vector in E^{n+1}. This provides a natural identification between $C_n(T)^*$ and E^{n+1}. In particular, the point functional, $L_t(f) \equiv f(t)$, $\forall\, f \in C_n(T)$, is represented in E^{n+1} by the vector $\boldsymbol{\varphi}(t) \equiv (\varphi_0(t), \varphi_1(t), \ldots, \varphi_n(t))^t$. Let $F \equiv \{\boldsymbol{\varphi}(t) \mid t \in T\} \subset E^{n+1}$ designate the set of point functionals. By continuity, F is a compact set.

[1] This work was partially supported by the Office of Naval Research, Contract Nonr 562(36).

A linear functional L is said to be *strictly positive* if $p(t) \geqslant 0$, $\forall\, t \in T$, $p(t) \not\equiv 0 \Rightarrow L(p) > 0$, and *nonnegative* if $p(t) \geqslant 0$, $\forall\, t \in T \Rightarrow L(p) \geqslant 0$. A nonnegative functional which is not strictly positive is said to be *singularly positive*. The convex cone of nonnegative linear functionals is denoted by M_n (either as a set in E^{n+1} with moment vectors as elements, or as a set in $C_n(T)^*$, the dual space, with functionals as elements). Clearly, $F \subset M_n$.

For any set C, let $K(C)$ designate the cone hull of C (i.e., the smallest convex cone with vertex at the origin which contains C), and let $H(C)$ designate the convex hull of C. Then, the following facts are known (Rogosinski ([8], [9], Wilson [11]):

(i) M_n is a closed, convex, pointed cone in E^{n+1}, with non-empty interior.
(ii) $M_n = K(F) = K(H(F))$.
(iii) $M_n{}^0$ (the interior of M_n) is the set of strictly positive linear functionals, while ∂M_n (the boundary of M_n) is the set of singularly positive linear functionals.
(iv) For $L \in M_n$, $\exists$ points $t_0, t_1, \ldots, t_p \in T$, and *positive* scalars $\lambda_0, \lambda_1, \ldots, \lambda_p$, such that

$$L(f) = \sum_{i=0}^{p} \lambda_i f(t_i), \qquad \forall\, f \in C_n(T),$$

where $p \leqslant n$, and such that the set of vectors $\{\boldsymbol{\varphi}(t_0), \boldsymbol{\varphi}(t_1), \ldots, \boldsymbol{\varphi}(t_p)\}$ is linearly independent in E^{n+1}.

In (i) a cone is *pointed* if the only linear subspace it contains is the set $\{0\}$.

It is easily observed that F spans the space E^{n+1}. Suppose not. Then, F is contained in a hyperplane through the origin. Thus, $\exists\, \mathbf{h} \neq 0$ such that $F \subset \{\mathbf{x} \mid \mathbf{h} \cdot \mathbf{x} = 0\}$. Hence, for each $t \in T$, $\mathbf{h} \cdot \boldsymbol{\varphi}(t) = 0$, so the "polynomial" $b(t) \equiv \mathbf{h} \cdot \boldsymbol{\varphi}(t)$ is identically zero, contradicting the linear independence of $\varphi_0, \varphi_1, \ldots, \varphi_n$ on T. By means of this observation, (iv) (an immediate consequence of (ii)) can be strengthened to

THEOREM 2.1. *Let L be a nonnegative linear functional on $C_n(T)$. Then, $\exists$ points $t_0, t_1, \ldots, t_n$, and nonnegative scalars $\lambda_0, \lambda_1, \ldots, \lambda_n$, such that*

(i) $$L(f) = \sum_{i=0}^{n} \lambda_i f(t_i), \qquad \forall\, f \in C_n(T), \tag{2.1}$$

(ii) *the set of vectors $\{\boldsymbol{\varphi}(t_0), \boldsymbol{\varphi}(t_1), \ldots, \boldsymbol{\varphi}(t_n)\}$ is linearly independent in E^{n+1}.*

Proof. From (iv), we have

$$L(f) = \sum_{i=0}^{p} \lambda_i f(t_i), \qquad \forall\, f \in C_n(T),$$

where $\lambda_i > 0$. Since F spans E^{n+1}, the linearly independent set $\{\boldsymbol{\varphi}(t_0), \ldots, \boldsymbol{\varphi}(t_p)\}$ can be augmented by elements of F, $\boldsymbol{\varphi}(t_{p+1}), \ldots, \boldsymbol{\varphi}(t_n)$, to provide a basis for E^{n+1}. Define $\lambda_{p+1} = \lambda_{p+2} = \ldots = \lambda_n = 0$. Hence,

$$L(f) = \sum_{i=0}^{n} \lambda_i f(t_i), \qquad \forall f \in C_n(T). \qquad \text{Q.E.D.}$$

Since the equations (2.1) are equivalent to the system of equations

$$\sum_{j=0}^{n} \lambda_j \varphi_i(t_j) = m_i \qquad i = 0, 1, \ldots, n, \tag{2.2}$$

where $m_i = L(\varphi_i)$, (ii) implies that the matrix of the system is nonsingular, and that the coefficients $\lambda_0, \lambda_1, \ldots, \lambda_n$ are determined uniquely. Thus, with this set of points, we can find a solution to the system

$$\sum_{j=0}^{n} \mu_j \varphi_i(t_j) = c_i, \qquad i = 0, 1, 2, \ldots, n,$$

say, $\boldsymbol{\mu}(\mathbf{c})$. Clearly, $\boldsymbol{\mu}(\mathbf{c})$ is continuous in $\mathbf{c}$, and $\boldsymbol{\mu}(\mathbf{m}) = \lambda \geqslant 0$.

This means that there exists a (unique) expression of the form (2.1) for any linear functional M on $C_n(T)$ involving the points $t_0, t_1, \ldots, t_n$ determined by L (although the coefficients λ_i may have any sign). Thus, we say the points $t_0, t_1, \ldots, t_n$ can be used to interpolate to any linear functional.

We can prove a stronger result:

THEOREM 2.2. *Let Ω be a compact set in E^m. Let $\mathbf{c}(s)$, $s \in \Omega$, be a continuous map of Ω into $M_n{}^0$. Then, $\exists$ points $t_0, t_1, \ldots, t_N$ in T, and functions $\lambda_0(s), \lambda_1(s), \ldots, \lambda_N(s)$, nonnegative on Ω, such that*

$$\mathbf{c}(s) = \sum_{i=0}^{N} \lambda_i(s) \boldsymbol{\varphi}(t_i)$$

where, in general, $N \geqslant n$. Equivalently, if for each $s \in \Omega$, L_s is a strictly positive linear functional on $C_n(T)$ and L_s is a continuous map of Ω into the dual space $C_n(T)^*$, then, $\exists$ points $t_0, t_1, \ldots, t_N$ in T, and nonnegative functions $\lambda_0(s), \lambda_1(s), \ldots, \lambda_n(s)$, $s \in \Omega$, such that

$$L_s(f) = \sum_{i=0}^{N} \lambda_i(s) f(t_i), \qquad \forall f \in C_n(T).$$

Further, the set $\{\boldsymbol{\varphi}(t_0), \boldsymbol{\varphi}(t_1), \ldots, \boldsymbol{\varphi}(t_N)\}$ can be assumed to span E^{n+1}.

Proof. Let $C_\Omega \equiv \{\mathbf{c}(s) | s \in \Omega\}$. This is, by continuity, a compact subset of $M_n{}^0$. Hence $H(C_\Omega)$ is a convex, compact subset of $M_n{}^0$. By the standard approximation theorem for compact convex sets (Eggleston [5], p. 68), there exists a convex polytope Q, with vertices $\mathbf{P}_1, \mathbf{P}_2, \ldots, \mathbf{P}_m$, such that

$$C_\Omega \subset Q \subset M_n{}^0.$$

Now, each point of Q is a convex combination of its vertices (Q is the convex hull of its vertices, by definition), and each vertex can be expressed as in Theorem 2.1. Therefore, each point of C_Ω can be expressed as a nonnegative combination of a finite set of points of F. Q.E.D.

3. Interpolation to Functions Satisfying Elliptic Equations

Suppose D is an open, bounded, connected set in E^r, with boundary ∂D and closure $\bar{D}$. The differential operator

$$M[u] \equiv \sum_{i,j=1}^{r} a_{ij}(\mathbf{t}) \frac{\partial^2}{\partial t_i \partial t_j} u(\mathbf{t}) + \sum_{i=1}^{r} b_i(\mathbf{t}) \frac{\partial}{\partial t_i} u(\mathbf{t})$$

where $a_{ij} = a_{ji}$, $\mathbf{t} \in D$, is said to be *uniformly elliptic in D* if $\exists\, \mu > 0$, such that

$$\sum_{i,j=1}^{r} a_{ij}(\mathbf{t}) \xi_i \xi_j \geqslant \mu \sum_{i=1}^{r} \xi_i^2, \qquad \forall\, \boldsymbol{\xi} \in E^r, \forall\, \mathbf{t} \in D.$$

We will require the following theorem given in Protter and Weinberger [7], p. 64.

Theorem 3.1. *Let u satisfy in D the differential inequality $M[u] + hu \geqslant 0$, where M is uniformly elliptic in D, $h(\mathbf{t}) \leqslant 0$, and where h and the coefficients of M (a_{ij} and b_i) are bounded. If u attains a nonnegative maximum K at a point of D, then $u \equiv K$.*

Corollary 3.1.1. *Let u, continuous in $\bar{D}$, satisfy in D the differential inequality above, and suppose that $u(\mathbf{t}) \leqslant 0$, $\mathbf{t} \in \partial D$. Then $u(\mathbf{t}) < 0$ in D, unless $u \equiv 0$.*

Proof. Immediate from Theorem 3.1.

Let $v_0, v_1, \ldots, v_n$ be continuous functions on $\bar{D}$, which satisfy in D the differential equation

$$M[u] + hu = 0,$$

and such that, as functions defined on ∂D, the boundary, they are linearly independent. We denote their span by $C_n(\partial D)$ or $C_n(\bar{D})$, depending on which set, ∂D or $\bar{D}$, we are considering as their domain of definition.

Lemma 3.1. *Let $\mathbf{t}^* \in D$. The linear functional $L_{\mathbf{t}^*}(f) \equiv f(\mathbf{t}^*)$, $\forall\, f \in C_n(\partial D)$, is strictly positive.*

Proof. Suppose $v(\mathbf{t}) \in C_n(\partial D)$, $v(\mathbf{t}) \not\equiv 0$, $v(\mathbf{t}) \geqslant 0$. Let $u(\mathbf{t}) = -v(\mathbf{t})$. Then, $M[u] + hu = 0$, and by Corollary 3.1.1, $u(\mathbf{t}^*) < 0$. Hence, $v(\mathbf{t}^*) > 0$, and the functional is strictly positive. Q.E.D.

We now assume that $C_n(\partial D)$ or $C_n(\bar{D})$ satisfies the Krein condition (the existence of a function strictly positive on its domain of definition). These are equivalent assumptions, since $\partial D \subset \bar{D}$ and by the maximum principle (Theorem 3.1).

THEOREM 3.2. *Let* $\mathbf{t}^* \in D$. *Then, under the above conditions,* $\exists$ *points* $\mathbf{t}_0, \mathbf{t}_1, \ldots, \mathbf{t}_n \in \partial D$, *and nonnegative scalars* $\lambda_0, \lambda_1, \ldots, \lambda_n$, *such that*

$$v(\mathbf{t}^*) = \sum_{i=0}^{n} \lambda_i v(\mathbf{t}_i), \qquad \forall\, v \in C_n(\bar{D}).$$

Further, the points $\mathbf{t}_0, \mathbf{t}_1, \ldots, \mathbf{t}_n$ *can be used for interpolation. That is, for each* $\mathbf{t} \in \bar{D}$,

$$v(\mathbf{t}) = \sum_{i=0}^{n} \alpha_i(\mathbf{t})\, v(\mathbf{t}_i), \qquad \forall\, v \in C_n(\bar{D}),$$

where

$$\begin{bmatrix} \alpha_0(\mathbf{t}) \\ \\ \alpha_n(\mathbf{t}) \end{bmatrix} = \begin{bmatrix} v_0(\mathbf{t}_0), \ldots, v_0(\mathbf{t}_n) \\ \\ v_n(\mathbf{t}_0), \ldots, v_n(\mathbf{t}_n) \end{bmatrix}^{-1} \begin{bmatrix} v_0(\mathbf{t}) \\ \\ v_n(\mathbf{t}) \end{bmatrix} \tag{3.1}$$

for $\mathbf{t} = \mathbf{t}^*$, $\alpha_i(\mathbf{t}^*) = \lambda_i \geqslant 0$, $i = 0, 1, \ldots, n$.

Proof. This is implied by Theorem 2.1 and Lemma 3.1. Q.E.D.

From (3.1), we see that $\alpha_i(\mathbf{t})$ is in $C_n(\bar{D})$, and thus satisfies the differential equation. Further, it follows directly from (3.1) that $\alpha_i(\mathbf{t}_j) = \delta_{ij}$. There is, of course, no guarantee that $\alpha_i(\mathbf{t}) \geqslant 0$ as $\mathbf{t}$ varies in $\bar{D}$. However, the following may be said.

THEOREM 3.3. *Let G be a compact set contained in D. Then,* $\exists$ *points* $t_0, t_1, \ldots, t_N \in \partial D$, *and functions* $\lambda_0(\mathbf{t}), \lambda_1(\mathbf{t}), \ldots, \lambda_N(\mathbf{t})$, *such that*

$$v(\mathbf{t}) = \sum_{i=0}^{N} \lambda_i(\mathbf{t})\, v(\mathbf{t}_i), \qquad \begin{array}{l} \forall\, v \in C_n(\bar{D}), \\ \forall\, \mathbf{t} \in D, \end{array}$$

where $\lambda_i(\mathbf{t}) \geqslant 0$ *for* $\mathbf{t} \in G$. *(*$N \geqslant n$*, usually.)*

Proof. Theorem 2.2 and Lemma 3.1. Q.E.D.

Interpolation formulas of the type of Theorem 3.2 are, of course, special cases of the type of Theorem 3.3.

Let $\varphi_0(\mathbf{t}), \varphi_1(\mathbf{t}), \ldots$ be a sequence of solutions of the differential equation $M[u] + hu = 0$, which are continuous in $\bar{D}$, linearly independent as functions on ∂D, and where $\varphi_0(\mathbf{t}) \geqslant \alpha > 0$ on $\bar{D}$.

Consider, next, a sequence ($n = 0, 1, 2, \ldots$) of interpolation formulas

$$u_n(f; \mathbf{t}) = \sum_{i=0}^{N(n)} \lambda_i^{(n)}(\mathbf{t})\, f(\mathbf{t}_i^{(n)})$$

given by Theorem 3.3 (for Theorem 3.2, $N(n) = n$) and exact for $\varphi_0, \varphi_1, \ldots, \varphi_n$.

We have:

$$\varphi_0(\mathbf{t}) = \sum_{i=0}^{N(n)} \lambda_i^{(n)}(\mathbf{t})\, \varphi_0(\mathbf{t}_i), \qquad \mathbf{t} \in D,$$

$$\geqslant \alpha \sum_{i=0}^{N(n)} \lambda_i^{(n)}(\mathbf{t}), \qquad \mathbf{t} \in G,$$

so that

(i) $$0 \leqslant \lambda_i^{(n)}(\mathbf{t}) = K/\alpha, \qquad \mathbf{t} \in G,$$

independently of n,

(ii) $$0 \leqslant \sum_{i=0}^{N(n)} |\lambda_i^{(n)}(\mathbf{t})| \leqslant K/\alpha, \qquad \mathbf{t} \in G,$$

independently of n,

where $K = \max\{\varphi_0(\mathbf{t}), \mathbf{t} \in G\}$.

Suppose H is a Banach space of functions on $\bar{D}$, each continuous there, and such that each satisfies the differential equation $M[u] + hu = 0$. (We assume that the norm in H is the sup norm.) Then, if $\varphi_0, \varphi_1, \ldots$ is a closed sequence in H,

$$u_n(f, \mathbf{t}) \to f(t), \qquad \text{as } n \to \infty$$

uniformly on G. This is an immediate consequence of the uniform boundedness principle (see Davis [*2*], p. 351).

This type of formula should be contrasted with the convergence of harmonic polynomials of interpolation using points defined by Curtiss for domains in the plane. (See Curtiss [*1*].) Curtiss exhibits the existence of interpolation schemes ($N(n) = n$) where uniform convergence occurs on compact subsets. In this paper we have exhibited schemes which converge at a point, or on a compact subset, specified in advance.

From a numerical point of view, the methods outlined in Wilson [*12*] can be used to obtain the interpolation formulas of the type of Theorem 3.2. We shall present some numerical results in a sequel.

References

1. J. H. Curtiss, Interpolation by harmonic polynomials. *SIAM J.* **10** (1962), 709–736.

2. P. J. Davis, "Interpolation and Approximation." Blaisdell, New York, 1963.

3. P. J. Davis, A construction of nonnegative approximate quadratures. *Math. Comp.* **21** (1967), 578–582.

4. P. J. Davis, Approximate integration rules with non-negative weights. "Lectures in Differential Equations," Vol. II. Van Nostrand, Princeton, 1968.

5. H. G. Eggleston, "Convexity." Cambridge University Press, Cambridge, 1958.

6. S. Karlin and W. J. Studden, "Tchebycheff Systems: with Applications in Analysis and Statistics." Wiley (Interscience), New York, 1966.

7. M. H. PROTTER AND H. F. WEINBERG, "Maximum Principles in Differential Equations." Prentice-Hall, New Jersey, 1967.
8. W. W. ROGOSINSKI, Moments of nonnegative mass. *Proc. Royal Soc. A,* **245** (1959), 1–27.
9. W. W. ROGOSINSKI, On nonnegative polynomials. *Ann. Univ. Sci. Budapest, Sect. Math.* **4** (1961), 253–280.
10. V. TCHAKALOFF, Formules de cubatures mécaniques à coefficients non négatifs. *Bull. Sci. Math.* **81** (1957), 123–134.
11. M. W. WILSON, Geometric aspects of quadratures with non-negative weights. Ph.D. Thesis, Brown University, 1968.
12. M. W. WILSON, A general algorithm for nonnegative quadrature formulas. Brown University Technical Report, June 1968.